

Black Holes and U-Duality

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Abstract

We find the general charged rotating black hole solutions of the maximal supergravities in dimensions $4 \leq D \leq 9$ arising from toroidally compactified Type II string or M-theories. In each dimension, these are obtained by acting on a generating solution with classical duality symmetries. In $D = 4$, $D = 5$ and $6 \leq D \leq 9$ the generating solution is specified by the ADM mass, $[\frac{D-1}{2}]$ -angular momentum components and five, three and two charges, respectively. We discuss the BPS-saturated (static) black holes and derive the U -duality invariant form of the area of the horizon. We also comment on the U -duality invariant form of the BPS mass formulae.

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I. INTRODUCTION

Black holes play an important role in string theory and recent developments (for a review, see [1]) have shown that string theory makes it possible to address their microscopic properties, in particular the statistical origin of the black hole entropy and possibly issues of information loss. The starting point in such investigations is the classical black hole solution and the aim of this paper is to find such solutions in toroidally compactified string theories in dimensions $4 \leq D \leq 9$.

The general solutions are found from a particular black hole “generating solution”, which is specified by a canonical choice of the asymptotic values of the scalar fields, the ADM mass, $[\frac{D-1}{2}]$ -components of angular momentum and a (minimal) number of charge parameters. The most general black hole, compatible with the “no-hair theorem”, is then obtained by acting on the generating solution with classical duality transformations. These are symmetries of the supergravity equations of motion, and so generate new solutions from old. They do not change the D -dimensional Einstein-frame metric but do change the charges and scalar fields. We first consider transformations, belonging to the maximal compact subgroup of duality transformations, which preserve the canonical asymptotic values of the scalar fields and show that all charges are generated in this way. Another duality transformation can be used to change the asymptotic values of the scalar fields.

For the toroidally compactified heterotic string such a program is now close to completion. Particular examples of solutions had been obtained in a number of papers (for a recent review and references, see [1]). In dimensions $D = 4$, $D = 5$ and $6 \leq D \leq 9$ the generating solution has five, three and two charge parameters, respectively. The charge parameters of the generating solution are associated with the $U(1)$ gauge fields arising from Kaluza-Klein (momentum modes) and two-form (winding modes) sectors for at most two toroidally compactified directions. The general black hole solution is then obtained by applying to the generating solution a subset of transformations, belonging to the maximal compact subgroup of the T - and S -duality transformations [2]. The explicit expression for the generating solution has been obtained in $D = 5$ [3] and $D \geq 6$ [4,5], however, in $D = 4$ only the five charge static generating solution [6] (see also [7]) and the four charge rotating solutions [8] were obtained.

The BPS-saturated solutions of the toroidally compactified heterotic string have non-singular horizons only for $D = 4$ static black holes [9,10] and $D = 5$ black holes with one non-zero angular momentum component [11]. In $6 \leq D \leq 9$ the BPS-saturated black holes have singular horizons with zero area. The explicit T - and S -duality invariant formulae for the area of the horizon and the ADM mass for the general BPS-saturated black holes were given for $D = 4$ in [9,10] and for $D = 5$ in [11]. In particular, the area of the horizon of the BPS-saturated black holes *does not* depend on the asymptotic values of the scalar fields [12,10,13,14], suggesting a microscopic interpretation.

The purpose of this paper is to study properties of the classical black hole solutions of *toroidally compactified Type II string theory or M-theory*, in dimensions $4 \leq D \leq 9$, thus completing the program for the toroidally compactified superstring vacua. We identify the minimum number of charge parameters for the generating solutions, which fully specifies the space-time metric of the *general* black hole in D -dimensions. The “toroidally” compactified sector of the heterotic string and the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of the

toroidally compactified Type II string have the *same* effective action and so have the same classical solutions. In this paper we shall show that the generating solutions for black holes in the toroidally compactified Type II string theory (or M-theory) are the *same* as the ones of toroidally compactified heterotic string. Note that it could have been the case that a more general generating solution with one or more RR charges was needed. Applying U -duality transformations to the generating solution generates all black holes of toroidally compactified Type II string theory (or M-theory).

We further address the BPS-saturated solutions, identify the U -duality invariant expression for the area of the horizon, i.e. Bekenstein-Hawking (BH) entropy for the general BPS-saturated black holes and outline a procedure to obtain the manifestly U -duality invariant mass formulae.

The paper is organized as follows. In Section II we summarize the symmetries of the effective action of toroidally compactified Type II string and obtain the general solution by applying a compact subgroup of duality transformations to the generating solution. In Section III we concentrate on general static BPS-saturated black holes in $4 \leq D \leq 9$ and derive the U -duality invariant expression for the area of the horizon. In Appendix A the effective action of the NS-NS sector of toroidally compactified Type II string is given. The explicit form of some of the generating solutions is given in Appendix B.

II. TOROIDALLY COMPACTIFIED TYPE II STRING THEORY

A. Symmetries

The low-energy effective action for the Type II string or M-theory, toroidally compactified to D -dimensions, is the maximal supergravity theory, which has a continuous duality symmetry U of its equations of motion [15] (see Table I, first column). This has a maximal compact subgroup C_U (second column in Table I). In the quantum theory the continuous classical symmetry U is broken to a discrete subgroup Q_U [16] (third column in Table I) which is the U -duality symmetry of the string theory. However, we will sometimes refer to the group U as the classical U -duality.

B. Solution Generating Technique

The general black hole solution is obtained by acting on generating solutions with U -duality transformations.

The scalar fields take values in the coset U/C_U and can be parameterised by a U -valued matrix $\mathcal{V}(x)$ which transforms under rigid U -transformations from the right and local C_U transformations from the left [15]. The Kaluza-Klein and antisymmetric tensor $U(1)$ gauge fields also transform under U . It is convenient to define $\mathcal{M} = \mathcal{V}^t \mathcal{V}$ which is inert under C_U and transforms under U as $\mathcal{M} \rightarrow \Omega \mathcal{M} \Omega^T$ ($\Omega \in U$).

The asymptotic value \mathcal{M}_∞ of \mathcal{M} can be brought to the canonical value $\mathcal{M}_{0\infty} = \mathbf{1}$ by a suitable U -duality transformation Ω_0 . The canonical value $\mathcal{M}_{0\infty}$ is preserved by C_U and the most general solution with the asymptotic behaviour $\mathcal{M}_\infty = \mathcal{M}_{0\infty}$ is obtained by acting on the generating solution with a subset of C_U transformations, i.e. the C_U orbits which are of

the form C_U/C_0 where C_0 is the subgroup preserving the generating solution. In particular, with this procedure the complete set of charges is obtained. Indeed, the generating solution is labelled by n_0 charges ($n_0 = 5, 3, 2$ for $D = 4, 5, \geq 6$, respectively) and if the dimension of the C_U orbits is n_1 , then $n_0 + n_1$ is the correct dimension of the vector space of charges for the general solution, as we shall check in the following Section. Black holes with arbitrary asymptotic values of scalar fields \mathcal{M}_∞ can then be obtained from these by acting with Ω_0 .

We shall seek the general charged rotating black hole solutions. In $D = 4$, such solutions are specified by electric *and* magnetic charges, while in $D > 4$ they carry electric charges only (once all $(D - 3)$ -form gauge fields have been dualised to vector fields).

III. BLACK HOLES IN VARIOUS DIMENSIONS

We will first propose generating solutions for Type II string (or M-theory) black holes in dimensions $4 \leq D \leq 9$ and then go on to show that the action of duality transformations generates all solutions. Remarkably, the generating solutions are the same as those used for the heterotic string. We will discuss only the charge assignments here, and give the explicit solutions in Appendix B.

A. Charge Assignments for the Generating Solution

1. $D=4$

The generating solution is specified in terms of *five* charge parameters. It is convenient to choose these to arise in the NS-NS sector of the compactified Type II string as follows. We choose two of the toroidal dimensions labelled by $i = 1, 2$ and let $A_{\mu i}^{(1)}$ be the two graviphotons (corresponding to $G_{\mu i}$) and $A_{\mu i}^{(2)}$ the two $U(1)$ gauge fields coming from the antisymmetric tensor (corresponding to $B_{\mu i}$) (see Appendix A). Corresponding to these four $U(1)$ gauge fields there are four electric charges $Q_i^{(1),(2)}$ and four magnetic ones $P_i^{(1),(2)}$. The generating solution, however carries the following five charges: $Q_1 \equiv Q_1^{(1)}$, $Q_2 \equiv Q_1^{(2)}$, $P_1 \equiv P_2^{(1)}$, $P_2 \equiv P_2^{(2)}$ and $q \equiv Q_2^{(1)} = -Q_2^{(2)}$. It will be useful to define the left-moving and right-moving charges $Q_{iL,R} \equiv Q_i^{(1)} \mp Q_i^{(2)}$ and $P_{iL,R} \equiv P_i^{(1)} \mp P_i^{(2)}$ ($i = 1, 2$). The generating solution then carries five charges associated with the first two compactified toroidal directions of the NS-NS sector, where the dyonic charges are subject to the constraint $\vec{P}_R \vec{Q}_R = 0$. We choose the convention that all the five charge parameters are positive.

2. $D=5$

In $D = 5$ the generating solution is parameterised by three (electric) charge parameters: $Q_1 \equiv Q_1^{(1)}$, $Q_2 \equiv Q_1^{(2)}$, and \tilde{Q} . Here the electric charges $Q_i^{(1),(2)}$ arise respectively from the graviphoton $A_{\mu i}^{(1)}$ and antisymmetric tensor $A_{\mu i}^{(2)}$ $U(1)$ gauge fields of the i -th toroidally compactified direction of the NS-NS sector, and \tilde{Q} is the electric charge of the gauge field, whose field strength is related to the field strength of the two-form field $B_{\mu\nu}$ by duality

transformation (see Appendix A). Again we choose the convention that all three charges are positive.

3. $6 \leq D \leq 9$

In $6 \leq D \leq 9$ the generating solution is parameterised by two electric charges: $Q_1 \equiv Q_1^{(1)}$, $Q_2 \equiv Q_1^{(2)}$. Again, the electric charges $Q_i^{(1),(2)}$ arise respectively from the graviphoton $A_{\mu i}^{(1)}$ and antisymmetric tensor $A_{\mu i}^{(2)}$ $U(1)$ gauge fields of the i -th toroidally compactified direction and we use the convention that both charges are positive.

Note that the explicit form of the generating solutions with the above charge assignments is the *same* as the one of the toroidally compactified heterotic string, since the corresponding NS-NS sector of the toroidally compactified string and the “toroidal” sector of the heterotic string are the same.

B. Action of Duality Transformations on Generating Solution

1. $D=4$

The $N = 8$ supergravity has 28 abelian gauge fields and so the general black hole solution carries 56 charges (28 electric and 28 magnetic). The U -duality group is $E_{7(7)}$, the maximal compact subgroup C_U is $SU(8)$ and the T -duality subgroup is $SO(6,6)$. We use the formulation with rigid E_7 symmetry and local $SU(8)$ symmetry [15]. The 56 charges fit into a vector \mathcal{Z} transforming as a **56** of E_7 . In the quantum theory, \mathcal{Z} is constrained to lie in a lattice by charge quantisation [16]. This “bare” charge vector can be “dressed” with the asymptotic value \mathcal{V}_∞ of the scalar field matrix \mathcal{V} to form

$$\bar{\mathcal{Z}} = \mathcal{V}_\infty \mathcal{Z} = \begin{pmatrix} q^{ab} \\ p_{ab} \end{pmatrix}, \quad (1)$$

which is invariant under E_7 but transforms under local $SU(8)$. The 28 electric and 28 magnetic dressed charges are q_{ab} and p_{ab} ($a, b = 1, \dots, 8$ and $q_{ab} = -q_{ba}, p_{ab} = -p_{ba}$). They can be combined to form the Z_{4AB} matrix ($A, B = 1, \dots, 8$ are $SU(8)$ indices) transforming as the complex antisymmetric representation of $SU(8)$, by defining $Z_{4AB} = (q^{ab} + ip_{ab})(\Gamma^{ab})^B_A$ where $(\Gamma^{ab})^B_A$ are the generators of $SO(8)$ in the spinor representation [15]. The matrix Z_{4AB} appears on the right hand side of the anticommutator of chiral two-component supercharges

$$[Q_{A\alpha}, Q_{B\beta}] = C_{\alpha\beta} Z_{4AB}, \quad (2)$$

and thus corresponds to the matrix of 28 complex central charges. An $SU(8)$ transformation $Z_4 \rightarrow Z_4^0 = (\mathcal{U} Z_4 \mathcal{U}^T)$ brings this charge matrix to the skew-diagonal form:

$$Z_4^0 = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_4 & 0 \end{pmatrix}, \quad (3)$$

where the complex λ_i ($i = 1, 2, 3, 4$) are the skew eigenvalues.

For the generating solution with the five charge parameters $Q_{1,2}$, $P_{1,2}$ and q (see Subsection III A) the eigenvalues are ¹:

$$\begin{aligned} \lambda_1 &= Q_{1R} + P_{2R}, \\ \lambda_2 &= Q_{1R} - P_{2R}, \\ \lambda_3 &= Q_{1L} + P_{2L} + 2iq, \\ \lambda_4 &= Q_{1L} - P_{2L} - 2iq \end{aligned} \quad (4)$$

(recall $Q_{1L,R} \equiv Q_1 \mp Q_2$ and $P_{2L,R} \equiv P_1 \mp P_2$).

We now consider the action of duality transformations on the generating solution and show that all $D = 4$ black hole solutions are indeed generated. The U -duality group E_7 has a maximal subgroup $SO(6, 6) \times SL(2, \mathbf{R})$ where $SO(6, 6)$ is the T -duality group and $SL(2, \mathbf{R})$ is the S -duality group. (Strictly speaking, the duality groups are discrete subgroups of these.) The **56** representation of E_7 decomposes as

$$\mathbf{56} \rightarrow (12, 2) \oplus (32, 1) \quad (5)$$

under $SO(6, 6) \times SL(2, \mathbf{R})$ and thus the 56 charges \mathcal{Z} decompose into 12 electric and 12 magnetic charges in the NS-NS sector, and 32 charges in the Ramond-Ramond (RR) sector. We choose for now the asymptotic value of the scalars to be the canonical one, i.e. $\mathcal{V}_\infty = \mathcal{V}_{0\infty} \equiv \mathbf{1}$.

The maximal compact symmetry of the T -duality group is $SO(6)_L \times SO(6)_R \sim SU(4)_L \times SU(4)_R$, and under $SU(4)_L \times SU(4)_R \subset SU(8)$ the complex representation **28** decomposes into complex representations $(12, 1) + (1, 12) + (4, 4)$. This decomposition corresponds to splitting the 8×8 matrix of charges Z_4 into 4×4 blocks. The two 4×4 diagonal blocks Z_R and Z_L , transform respectively as the antisymmetric complex representations of $SU(4)_{R,L} \sim SO(6)_{R,L}$ and represent the 12 + 12 charges of the NS-NS sector. The off-diagonal blocks correspond to the 16 complex RR charges.

The maximal compact subgroup of $SO(6, 6) \times SL(2, \mathbf{R})$ is $SO(6)_L \times SO(6)_R \times SO(2)$ and it preserves $\mathcal{V}_{0\infty}$. The subgroup that preserves the charges of the generating solution is $SO(4)_L \times SO(4)_R$. Thus acting on the generating solution with $SO(6)_L \times SO(6)_R \times SO(2)$, gives orbits corresponding to the 19-dimensional space

¹For the four charges $Q_{1,2}$ and $P_{1,2}$ the eigenvalues were given in [17].

$$\frac{SO(6)_L \times SO(6)_R}{SO(4)_L \times SO(4)_R} \times SO(2) . \quad (6)$$

As the generating solution has five charges, acting on the generating solution with $SO(6)_L \times SO(6)_R \times SO(2)$ gives the required $5 + 19 = 24$ NS-NS charges, i.e. the 24 NS-NS charges are parameterised in terms of the five charges of the generating solution and the 19 coordinates of the orbit space (6).

The above procedure is closely related to that for $D = 4$ toroidally compactified heterotic string vacua [9,10], where the general black hole with the $5 + 51 = 56$ charges is obtained from the same five-parameter generating solution, and the 51 coordinates of the orbit

$$\frac{SO(22)_L \times SO(6)_R}{SO(20)_L \times SO(4)_R} \times SO(2) . \quad (7)$$

We can now generalise this procedure to include the RR charges. The group $C_U = SU(8)$ preserves the canonical asymptotic values of the scalar fields and only the subgroup $SO(4)_L \times SO(4)_R$ leaves the generating solution invariant. Then acting with $SU(8)$ gives orbits

$$SU(8)/[SO(4)_L \times SO(4)_R] \quad (8)$$

of dimension $63 - 6 - 6 = 51$. The $SU(8)$ action then induces 51 new charge parameters, which along with the original five parameters provide charge parameters for the general solution with 56 charges. Finally, the general black hole with arbitrary asymptotic values of the scalars is obtained from these 56-parameter solutions by acting with a E_7 transformation. This transformation leaves the central charge matrix Z_4 and its eigenvalues λ_i invariant, but changes the asymptotic values of scalars and “dresses” the physical charges. The orbits under E_7 are the 70-dimensional coset $E_7/SU(8)$, as expected.

The fact that the same five-parameter generating solution that was used for the $D = 4$ toroidally compactified heterotic string should be sufficient to generate all black holes with NS-NS charges of $D = 4$ toroidally compactified Type II is unsurprising, given the equivalence between the “toroidal” sector of the heterotic string and the NS-NS sector of the Type II string. However, it is interesting that the procedure outlined above is also sufficient to generate *all* RR charges of the general black hole solution, as it could have been the case that a more general generating solution carrying one or more RR charges was needed.

2. $D=5$

The U -duality group is $E_{6(6)}$, the maximal compact subgroup C_U is $USp(8)$ and the T -duality group is $SO(5,5)$ with its maximal compact subgroup $SO(5)_L \times SO(5)_R$. In this case there are 27 abelian gauge fields and the 27 electric charges (dressed with asymptotic values of the scalar fields) transform as a **27** of $USp(8)$ and can be assembled into an 8×8 matrix Z_{5AB} ($A, B = 1, \dots, 8$) with the properties [18]:

$$Z_5^{AB*} = \Omega^{AC} \Omega^{BD} Z_{5CD}, \quad \Omega^{AB} Z_{5AB} = 0, \quad (9)$$

where Ω is the $USp(8)$ symplectic invariant, which we take to be

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad (10)$$

With Ω given by (10), the Z_5 charge matrix can be written in the following form:

$$Z_5 = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{14} & \cdots \\ -z_{12} & 0 & -z_{14}^* & z_{13}^* & \cdots \\ -z_{13}^* & z_{14} & 0 & z_{34} & \cdots \\ -z_{14}^* & -z_{13} & -z_{34} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \quad (11)$$

Here z_{12} , z_{34} , z_{56} are real and satisfy $z_{12} + z_{34} + z_{56} = 0$.

The matrix Z_5 occurs in the superalgebra and represents the 27 (real) central charges. It can be brought into a skew-diagonal form of the type (3) by an $USp(8)$ transformation $Z_5 \rightarrow Z_5^0 = (\mathcal{U}Z_5\mathcal{U}^T)$. The four real eigenvalues λ_i are subject to the constraint: $\sum_{i=1}^4 \lambda_i = 0$.

The generating solution is parameterised by three charges $Q_1 \equiv Q_1^{(1)}$, $Q_2 \equiv Q_1^{(2)}$ and \tilde{Q} (see Subsection III A). The four (constrained, real) eigenvalues λ_i are then

$$\begin{aligned} \lambda_1 &= \tilde{Q} + Q_{1R}, \\ \lambda_2 &= \tilde{Q} - Q_{1R}, \\ \lambda_3 &= -\tilde{Q} + Q_{1L}, \\ \lambda_4 &= -\tilde{Q} - Q_{1L}, \end{aligned} \quad (12)$$

where $Q_{1L,R} \equiv Q_1 \mp Q_2$. These indeed satisfy the constraint $\sum_{i=1}^4 \lambda_i = 0$.

The three parameter solution is indeed the generating solution for black holes in $D = 5$. The $USp(8)$ duality transformations preserve the canonical asymptotic values of the scalar fields and the subgroup $SO(4)_L \times SO(4)_R [\subset SO(5)_L \times SO(5)_R] \subset USp(8)$ preserves the generating solution. Acting with $USp(8)$ on the generating solution gives orbits

$$USp(8)/[SO(4)_L \times SO(4)_R], \quad (13)$$

and thus induces $36 - 4 \times 3 = 24$ new charge parameters, which along with the original three charge parameters provide 27 electric charges for the general solution in $D = 5$.

3. $D=6$

The U -duality group is $SO(5, 5)$, the maximal compact subgroup C_U is $SO(5) \times SO(5)$ and the T -duality group is $SO(4, 4)$ which has maximal compact subgroup $SO(4)_L \times$

$SO(4)_R \sim [SU(2) \times SU(2)]_L \times [SU(2) \times SU(2)]_R$. There are 16 abelian vector fields and the bare charges \mathcal{Z} transform as a **16** (spinor) of $SO(5, 5)$. The dressed charges transform as the $(4, 4)$ representation of $SO(5) \times SO(5)$ and can be arranged into a 4×4 charge matrix Z_6 .

Under $[SU(2) \times SU(2)]_L \times [SU(2) \times SU(2)]_R \subset SO(5) \times SO(5)$ the $(4, 4)$ decomposes into $(2, 2, 1, 1) + (1, 1, 2, 2) + (1, 2, 2, 1) + (2, 1, 1, 2)$. This decomposition corresponds to splitting the 4×4 matrix of charges Z_6 into 2×2 blocks. The two 2×2 diagonal blocks Z_R and Z_L , transform respectively as $(2, 2, 1, 1)$ and $(1, 1, 2, 2)$ representations of $[SU(2) \times SU(2)]_L \times [SU(2) \times SU(2)]_R$ representing the $4+4$ charges of the NS-NS sector. The off-diagonal blocks correspond to $(1, 2, 2, 1) + (2, 1, 1, 2)$ representations of $[SU(2) \times SU(2)]_L \times [SU(2) \times SU(2)]_R$ and represent 8 RR charges.

The matrix Z_6 occurs in the superalgebra and represents the 16 (real) central charges. It can be brought into a skew-diagonal form of the type (3) by an $SO(5) \times SO(5)$ transformation $Z_6 \rightarrow Z_6^0 = (\mathcal{U} Z_6 \mathcal{U}^T)$ with the two eigenvalues λ_i . The generating solution is parameterised by two charges $Q_1 \equiv Q_1^{(1)}$, $Q_2 \equiv Q_1^{(2)}$ (see Subsection III A). The two eigenvalues λ_i are then

$$\begin{aligned}\lambda_1 &= Q_{1R}, \\ \lambda_2 &= Q_{1L},\end{aligned}\tag{14}$$

where again $Q_{1L,R} \equiv Q_1 \mp Q_2$.

The generating solution is preserved by $SO(3)_L \times SO(3)_R [\subset SO(4)_L \times SO(4)_R] \subset SO(5) \times SO(5)$ so acting with $SO(5) \times SO(5)$ gives

$$[SO(5) \times SO(5)]/[SO(3)_L \times SO(3)_R]\tag{15}$$

orbits, and thus introduces $2(10 - 3) = 14$ charge parameters, which along with the two charges $(Q_{1,2})$ of the generating solution provide the 16 charge parameters of the general solution in $D = 6$.

4. $D=7$

The U -duality group is $SL(5, \mathbf{R})$, the maximal compact subgroup C_U is $SO(5)$ and the T -duality group is $SO(3, 3)$ with its maximal compact subgroup $SO(3)_L \times SO(3)_R$. There are ten abelian vector fields and the ten bare electric charges transform as the **10** representation of $SL(5, \mathbf{R})$. Dressing of these with asymptotic values of scalars gives the ten central charges which are inert under $SL(5, \mathbf{R})$ but transform as a **10** under $SO(5)$. The dressed charges can be assembled into a real antisymmetric 5×5 charge matrix Z_{7ij} , which appears in the superalgebra as the 4×4 central charge matrix $Z_{7AB} = \frac{1}{2} Z_{7ij} \gamma^{ij}_{AB}$ where γ^{ij}_{AB} are the generators of $SO(5)$ in the spinor **(4)** representation. The matrix Z_{7ij} has two real skew eigenvalues, λ_1, λ_2 , which for the generating solution correspond to the two charges $Q_{1L,R}$.

The subgroup $SO(2)_L \times SO(2)_R \subset SO(3)_L \times SO(3)_R \subset O(5)$ preserves the generating solution, so that the action of $SO(5)$ gives orbits

$$SO(5)/[SO(2)_L \times SO(2)_R],\tag{16}$$

thus introducing $10 - 2 = 8$ charge parameters, which together with the two charges ($Q_{1,2}$) of the generating solution (see Subsection III A) provide the ten charges of the general solution in $D = 7$.

5. $D=8$

The U -duality group is $SL(3, \mathbf{R}) \times SL(2, \mathbf{R})$, the maximal compact subgroup C_U is $SO(3) \times U(1)$ and the T -duality group is $SO(2, 2)$ with maximal compact subgroup $SO(2)_L \times SO(2)_R$. There are six abelian gauge fields and the six bare electric charges transform as $(\mathbf{3}, \mathbf{2})$ under $SL(3, \mathbf{R}) \times SL(2, \mathbf{R})$.

No C_U transformations preserve the generating solution, so that the orbits are

$$C_U = SO(3) \times U(1) , \quad (17)$$

and the C_U transformations introduce $(3 + 1)$ -charge parameters, which along with the two charges of the generating solution provide the six charges of the general solution in $D = 8$.

6. $D=9$

The U -duality group is $SL(2, \mathbf{R}) \times \mathbf{R}^+$ and the maximal compact subgroup C_U is $U(1)$. There are three abelian gauge fields and the three bare electric charges transform as $(\mathbf{3}, \mathbf{1})$ under $SL(2, \mathbf{R}) \times \mathbf{R}^+$. The action of

$$C_U = U(1), \quad (18)$$

introduces one-charge parameter, which along with the two charges of the generating solution provides the three charges of the general solution in $D = 9$.

IV. ENTROPY AND MASS OF BPS-SATURATED STATIC BLACK HOLES

We now study the properties of static BPS-saturated solutions. In the preceding Section we identified the charge assignments for the generating solutions, which fully specify the space-time of the general black hole solution in D -dimensions for toroidally compactified Type II string (or M-theory) vacuum. The explicit form for these solutions has been given in the literature (with the exception of the rotating five-charge solution in $D = 4$). The static generating solutions are given in Appendix B. Also, the area of the horizon for the BPS-saturated (as well as for non-extreme solutions) was calculated explicitly. In addition to static BPS-saturated solutions, we shall also consider near-BPS-saturated solutions in $4 \leq D \leq 9$.

A. The Bogomol'nyi Bound

Consider first the $D = 4$ case. Standard arguments [19] imply that the ADM mass M is bounded below by the moduli of the eigenvalues λ_i (3) of the central charge matrix Z_4 , i.e. $M \geq |\lambda_i|$, $i = 1, \dots, 4$. Without loss of generality the eigenvalues can be ordered in such a way that $|\lambda_i| \geq |\lambda_j|$ for $j \geq i$.

If M is equal to $|\lambda_1| = \dots = |\lambda_p|$, the solution preserves $\frac{p}{8}$ of $N = 8$ supersymmetry². For example, if $M = |\lambda_1| > |\lambda_{2,3,4}|$ then $\frac{1}{8}$ of the supersymmetry is preserved, while for $M = |\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4|$, $\frac{1}{2}$ of the supersymmetry is preserved.

The eigenvalues λ_i are each invariant under E_7 and $SU(8)$. The physical quantities such as the Bekenstein-Hawking (BH) entropy and the ADM mass of BPS-saturated black holes, can then be written in terms of these quantities, which depend on both the bare charges \mathcal{Z} and the asymptotic values of the scalar fields parameterised by \mathcal{V}_∞ . However, there are special combinations of these invariants for which the dependence on the asymptotic values of scalar fields drops out. In particular, such combinations play a special role in the BH entropy for the BPS-saturated black hole solutions as discussed in the following Subsection. Similar comments apply in $D > 4$.

B. Bekenstein-Hawking Entropy

The BH entropy is defined as $S_{BH} = \frac{1}{4G_D} A_h$ where G_D is the D -dimensional Newton's constant and A_h is the area of the horizon. Since the Einstein metric is duality invariant, geometrical quantities such as A_h should be too. Thus it should be possible to write A_h in terms of duality invariant quantities such as the eigenvalues λ_i , the ADM mass (or the “non-extremality” parameter m defined in Appendix B) and the angular momentum components. However, in the case of BPS-saturated black holes the BH entropy is, in addition, *independent* of the asymptotic values of scalar fields. This property was pointed out in [12], and further exhibited for the general BPS-saturated solutions of toroidally compactified heterotic string vacua in $D = 4$ [10] and $D = 5$ [3] as well as for certain BPS-saturated black holes of $D = 4$ $N = 2$ superstring vacua in [21,13]. This property was explained in terms of “supersymmetric attractors” in [14].

When applying the above arguments to BPS-saturated black holes of toroidally compactified Type II string (or M-theory) their BH entropy can be written in terms of a U -duality invariant combination of *bare* charges alone, thus implying that only a very special combination of bare charges can appear in the BH entropy formula. One is then led to the remarkable result that the entropy must be given in terms of the quartic invariant of E_7 in $D = 4$ and the cubic invariant of E_6 in $D = 5$, as these are the only possible U -invariants of bare charges.³ This fact was first pointed out in [17] and [14], respectively (and indepen-

²For a related discussion of the number of preserved supersymmetries see [20,17].

³In $D = 3$ there is a unique quintic $E_{8(8)}$ -invariant which should play a similar role for $D = 3$ black hole solutions.

dently in [22]). It has been checked explicitly for certain classes of $D = 4$ BPS-saturated black holes [17]. Below we will extend the analysis to general BPS-saturated black holes in $D = 4, 5$.

For $6 \leq D \leq 9$ there are *no* non-trivial U -invariant quantities that can be constructed from the bare charges alone, in agreement with the result that there are no BPS-saturated black holes with non-singular horizons and finite BH entropy in $D \geq 6$, as has been shown explicitly in [23,4].

1. $D=4$

The five-parameter static generating solution has the following BH entropy [10]:

$$S_{BH} = 2\pi\sqrt{Q_1Q_2P_1P_2 - \frac{1}{4}q^2(P_1 + P_2)^2}. \quad (19)$$

We shall now show that (19) can be rewritten in terms of the E_7 quartic invariant (of bare charges).

The quartic $E_{7(7)}$ invariant J_4 , constructed from the charge matrix Z_{4AB} , is [15]:

$$J_4 = \text{Tr}(Z_4^\dagger Z_4)^2 - \frac{1}{4}(\text{Tr} Z_4^\dagger Z_4)^2 + \frac{1}{96}(\epsilon_{ABCDEFGH} Z_4^{AB*} Z_4^{CD*} Z_4^{EF*} Z_4^{GH*} + \epsilon^{ABCDEFGH} Z_{4AB} Z_{4CD} Z_{4EF} Z_{4GH}), \quad (20)$$

which can be written in terms of the skew-eigenvalues λ_i by substituting the skew-diagonalised matrix Z_4^0 (3) in (20) to give (as in [17]):

$$J_4 = \sum_{i=1}^4 |\lambda_i|^4 - 2 \sum_{j>i=1}^4 |\lambda_i|^2 |\lambda_j|^2 + 4(\lambda_1^* \lambda_2^* \lambda_3^* \lambda_4^* + \lambda_1 \lambda_2 \lambda_3 \lambda_4). \quad (21)$$

For the five parameter generating solution, the λ_i are given by (4), so that (21) becomes:

$$\begin{aligned} J_4 &= 16[(Q_{1R}^2 - Q_{1L}^2)(P_{2R}^2 - P_{2L}^2) - 4P_{2R}^2 q^2] \\ &= 16^2[Q_1 Q_2 P_1 P_2 - \frac{1}{4}q^2(P_1 + P_2)^2]. \end{aligned} \quad (22)$$

Comparing with (19), we learn that for the five-parameter generating solution the BH entropy is given by

$$S_{BH} = \frac{\pi}{8}\sqrt{J_4}. \quad (23)$$

This result generalises the one in [17], where the result for the four-parameter solution with $q = 0$ was established.

Acting on the generating solution with $SU(8)$ transformations to generate the general charged black hole, and then with a E_7 transformation to generate the solution with general asymptotic values of scalar fields, leaves the BH entropy (23) invariant, since J_4 is invariant. As the dressing of the charges is by an E_7 transformation, i.e. $\tilde{\mathcal{Z}} = \mathcal{V}_\infty \mathcal{Z}$, the dependence on the asymptotic values of scalar fields \mathcal{V}_∞ drops out of the BH entropy, which thus can be written in terms of the bare charges alone, as expected.

2. $D=5$

The BH entropy of the three-parameter static BPS-saturated generating solution is [24,11,3]:

$$S_{BH} = 2\pi\sqrt{Q_1 Q_2 \tilde{Q}}. \quad (24)$$

The cubic $E_{6(6)}$ invariant J_3 constructed from the charge matrix Z_{5AB} is [18]:

$$J_3 = - \sum_{A,B,C,D,E,F=1}^8 \Omega^{AB} Z_{5BC} \Omega^{CD} Z_{5DE} \Omega^{EF} Z_{5FA}. \quad (25)$$

By transforming Z_5 to a skew-diagonal matrix Z_5^0 , given in terms of the four constrained real eigenvalues λ_i (12), and using (10) for Ω_{AB} , the cubic form J_3 can be written as

$$J_3 = 2 \sum_{i=1}^4 \lambda_i^3. \quad (26)$$

For the (three charge parameter) generating solution the eigenvalues are (12) so that (26) is:

$$\begin{aligned} J_3 &= 12(Q_{1R}^2 - Q_{1L}^2)\tilde{Q} \\ &= 48Q_1 Q_2 \tilde{Q}, \end{aligned} \quad (27)$$

which together with (24) implies

$$S_{BH} = \pi\sqrt{\frac{1}{12}J_3}. \quad (28)$$

This result gives the entropy in terms of the cubic invariant for the generating solution, and so, by U -invariance, for all charged static BPS-saturated $D = 5$ black holes, as conjectured in [14,25,22].

3. $6 \leq D \leq 9$

A BPS-saturated black hole in $D \geq 6$ dimensions should have a horizon area that is an invariant under U -duality constructed from the bare charges alone, involving no scalars. This would involve, for example, constructing an $SO(5,5)$ singlet from tensor products of charges transforming as a chiral spinor **16** of $SO(5,5)$ in $D = 6$ and constructing singlets of $SL(5, \mathbf{R})$ from tensor products of charges transforming as a **10**. There are no such non-trivial singlets in $D \geq 6$, so that the only invariant result for the area is zero, which is precisely what is found (see Appendix B). Indeed, the generating solution for BPS-saturated solution in $D \geq 6$ has *zero* horizon area [4] and so zero BH entropy.

4. Entropy of Non-Extreme Black Holes

We now comment on the BH entropy of non-extreme black holes, in particular, static black holes in $6 \leq D \leq 9$. (For another approach to address the BH entropy of non-extreme black holes, where additional auxiliary charges are introduced, see [26,27].) The non-extreme generating solutions are specified in terms of two electric charges $Q_{1,2}$ and a parameter m which measures the deviation from extremality, i.e. the BPS-saturated limit is reached when $m = 0$ while the charges $Q_{1,2}$ are kept constant (see Appendix B). The BH entropy is given by (63) in Appendix B, which in the near-BPS-saturated limit ($Q_{1,2} \gg m$) reduces to the form (64):

$$S_{BH} = 4\pi \sqrt{\frac{1}{(D-3)^2} Q_1 Q_2 (2m)^{\frac{2}{D-3}}}. \quad (29)$$

The BH entropy of the non-extreme black holes can also be rewritten in a manifestly duality-invariant manner. We demonstrate this for static near-extreme black holes in $D = 7$; examples of such black holes in other dimensions are similar. The 5×5 matrix of dressed charges Z_{7ij} transforms as an antisymmetric tensor under the local $SO(5)$ symmetry but is invariant under the rigid $SL(5, \mathbf{R})$ duality symmetry. (The central charges are given by the 4×4 matrix $Z_{7ij} \gamma^{ij}$ where γ^{ij} are the generators of $SO(5)$ in the 4-dimensional spinor representation.) The two skew eigenvalues λ_i of Z_{7ij} are given by $\lambda_i = Q_{R,L}$ and these are invariant under $SO(5) \times SL(5, \mathbf{R})$. The BH entropy in the near-extreme case is then

$$S_{BH} = \frac{\pi}{2} \sqrt{(|\lambda_1|^2 - |\lambda_2|^2)(2m)^{\frac{1}{2}}}, \quad (30)$$

which can be rewritten in a manifestly U -duality invariant form as

$$S_{BH} = \frac{\pi}{2} \left([2\text{tr}(Y_7^2) - \frac{1}{2}(\text{tr} Y_7)^2] m \right)^{\frac{1}{4}}. \quad (31)$$

Here $Y_7 \equiv Z_7^t Z_7$ and we have used the relationship: $\text{tr}(Y_7^m) = 2(-|\lambda_1|^2)^m + (-|\lambda_2|^2)^m$. Note that now the entropy does depend on the asymptotic values of the scalar fields.

C. ADM Masses and Supersymmetry Breaking

We now comment on a U -duality invariant form of the black hole mass formula for BPS-saturated black holes with different numbers of preserved supersymmetries. We shall derive such expressions in $D = 4$. Examples in other dimensions are similar.

As discussed in the beginning of this Section, in $D = 4$ the BPS-saturated black holes will preserve $\frac{p}{8}$ of the $N = 8$ supersymmetry if the BPS-saturated ADM mass M is equal to $|\lambda_1| = \dots = |\lambda_p|$ where λ_i ($i = 1, \dots, 4$) are eigenvalues (3) of central charge matrix Z_4 . Without loss of generality one can order the eigenvalues in such a way that $|\lambda_i| \geq |\lambda_j|$ for $j \geq i$. Note also that from (3)

$$\text{tr}(Y_4^m) = 2 \sum_{i=1}^4 (-|\lambda_i|^2)^m, \quad (32)$$

where $Y_4 \equiv Z_4^\dagger Z_4$ and $m = 1, \dots, 5 - p$.

1. $p=4$

These solutions preserve $\frac{1}{2}$ of $N = 8$ supersymmetry and

$$M = |\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4|. \quad (33)$$

Examples of such solutions are obtained from the generating solution with only one non-zero charge, e.g., those with only $Q_1 \neq 0$. The mass can be written in a U -invariant form as

$$M = -\frac{1}{8}trY_4. \quad (34)$$

2. $p=3$

These solutions preserve $\frac{3}{8}$ of $N = 8$ supersymmetry thus

$$M = |\lambda_1| = |\lambda_2| = |\lambda_3| > |\lambda_4|. \quad (35)$$

An example of such a generating solution corresponds to the case with $(Q_1, Q_2, P_1 = P_2) \neq 0$, while non-zero q is determined in terms of other nonzero charges as $q = \frac{1}{2}[(Q_{1R} + P_{2R})^2 - Q_{1L}^2]^{1/2}$. The U -duality invariant form of the BPS-saturated mass can now be written in terms of two invariants:

$$trY_4 = -6|\lambda_1|^2 - 2|\lambda_4|^2, \quad tr(Y_4^2) = 6|\lambda_1|^4 + 2|\lambda_4|^4 \quad (36)$$

as the (larger) root of the quadratic equation:

$$48M^4 + 12trY_4M^2 + (trY_4)^2 - 2tr(Y_4^2) = 0, \quad (37)$$

$$M^2 = -\frac{1}{8}trY_4 + \sqrt{\frac{1}{24}tr(Y_4^2) - \frac{1}{192}(trY_4)^2}. \quad (38)$$

3. $p=2$

These solutions preserve $\frac{1}{4}$ of $N = 8$ supersymmetry and have the mass

$$M = |\lambda_1| = |\lambda_2| > |\lambda_3| \geq |\lambda_4|. \quad (39)$$

An example of such a generating solution is the case with only $(Q_1, Q_2) \neq 0$. The general mass can be written in terms of the three invariants $tr(Y_4^m)$ for $m = 1, 2, 3$. The U -duality invariant expression for the BPS-saturated mass formula is then given by the (largest) root of a cubic equation; we do not give it explicitly here. Some simplification is obtained if $|\lambda_3| = |\lambda_4|$, in which case only two invariants and a quadratic equation are needed.

4. $p=1$

These solutions preserve $\frac{1}{8}$ of the $N = 8$ supersymmetry and the mass is

$$M = |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| . \quad (40)$$

Examples of such generating solutions are the case with only $(Q_1, Q_2, P_1) \neq 0$ and the generating solution with all the five charges non-zero is also in this class. The U -invariant mass can be written in terms of the four invariants $tr(Y_4^m)$ for $m = 1, 2, 3, 4$ and involves the (largest) root of a quartic equation so we do not give it explicitly here.

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V. APPENDIX A: EFFECTIVE ACTION OF THE NS-NS SECTOR OF TYPE II STRING ON TORI

For the sake of completeness we briefly summarize the form of the effective action of the NS-NS sector for the toroidally compactified Type II string in D -dimensions (see, e.g., [28]). The notation used is that of [8].

The compactification of the $(10 - D)$ -spatial coordinates on a $(10 - D)$ -torus is achieved by choosing the following abelian Kaluza-Klein Ansatz for the ten-dimensional metric

$$\hat{G}_{MN} = \begin{pmatrix} e^{a\varphi} g_{\mu\nu} + G_{mn} A_\mu^{(1)m} A_\nu^{(1)n} & A_\mu^{(1)m} G_{mn} \\ A_\nu^{(1)n} G_{mn} & G_{mn} \end{pmatrix}, \quad (41)$$

where $A_\mu^{(1)m}$ ($\mu = 0, 1, \dots, D - 1$; $m = 1, \dots, 10 - D$) are D -dimensional Kaluza-Klein $U(1)$ gauge fields, $\varphi \equiv \hat{\Phi} - \frac{1}{2} \ln \det G_{mn}$ is the D -dimensional dilaton field, and $a \equiv \frac{2}{D-2}$. Then, the effective action is specified by the following massless bosonic fields: the (Einstein-frame) graviton $g_{\mu\nu}$, the dilaton e^φ , $(20 - 2D)$ $U(1)$ gauge fields $\mathcal{A}_\mu^i \equiv (A_\mu^{(1)m}, A_{\mu m}^{(2)})$ defined as $A_{\mu m}^{(2)} \equiv \hat{B}_{\mu m} + \hat{B}_{mn} A_\mu^{(1)n}$, and the following symmetric $O(10 - D, 10 - D)$ matrix of the scalar fields (moduli):

$$M = \begin{pmatrix} G^{-1} & -G^{-1}C \\ -C^T G^{-1} & G + C^T G^{-1}C \end{pmatrix}, \quad (42)$$

where $G \equiv [\hat{G}_{mn}]$, $C \equiv [\hat{B}_{mn}]$ and are defined in terms of the internal parts of ten-dimensional fields. Then the NS-NS sector of the D -dimensional effective action takes the form:

$$\begin{aligned} \mathcal{L} = & \frac{1}{16\pi G_D} \sqrt{-g} [\mathcal{R}_g - \frac{1}{(D-2)} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu M L \partial_\nu M L) \\ & - \frac{1}{12} e^{-2a\varphi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} - \frac{1}{4} e^{-a\varphi} g^{\mu\mu'} g^{\nu\nu'} \mathcal{F}_{\mu\nu}^i (L M L)_{ij} \mathcal{F}_{\mu'\nu'}^j], \end{aligned} \quad (43)$$

where $g \equiv \det g_{\mu\nu}$, \mathcal{R}_g is the Ricci scalar of $g_{\mu\nu}$, and $\mathcal{F}_{\mu\nu}^i = \partial_\mu \mathcal{A}_\nu^i - \partial_\nu \mathcal{A}_\mu^i$ are the $U(1)^{20-2D}$ gauge field strengths and $H_{\mu\nu\rho} \equiv (\partial_\mu B_{\nu\rho} - \frac{1}{2} \mathcal{A}_\mu^i L_{ij} \mathcal{F}_{\nu\rho}^j) + \text{cyc. perms. of } \mu, \nu, \rho$ is the field strength of the two-form field $B_{\mu\nu}$.

The D -dimensional effective action (43) is invariant under the $O(10 - D, 10 - D)$ transformations (T -duality):

$$M \rightarrow \Omega M \Omega^T, \quad \mathcal{A}_\mu^i \rightarrow \Omega_{ij} \mathcal{A}_\mu^j, \quad g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad \varphi \rightarrow \varphi, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}, \quad (44)$$

where Ω is an $O(10 - D, 10 - D)$ invariant matrix, *i.e.*, with the following property:

$$\Omega^T L \Omega = L, \quad L = \begin{pmatrix} 0 & I_{10-D} \\ I_{10-D} & 0 \end{pmatrix}, \quad (45)$$

where I_n denotes the $n \times n$ identity matrix.

In $D = 4$ the field strength of the abelian gauge field is self-dual, *i.e.* $\tilde{\mathcal{F}}^{i\mu\nu} = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma}^i$, and thus the charged solutions are specified by the electric and magnetic charges.

In $D = 5$ the effective action is specified by the graviton, 26 scalar fields (25 moduli fields in the matrix M and the dilaton φ), 10 $U(1)$ gauge fields, and the field strength $H_{\mu\nu\rho}$ of the two-form field $B_{\mu\nu}$. By the duality transformation $H^{\mu\nu\rho} = -\frac{e^{4\varphi/3}}{2!\sqrt{-g}}\varepsilon^{\mu\nu\rho\lambda\sigma}\tilde{F}_{\lambda\sigma}$, $H_{\mu\nu\rho}$ can be related to the field strength $\tilde{F}_{\mu\nu}$ of the gauge field \tilde{A}_μ , which specifies an additional electric charge \tilde{Q} .

In $D \geq 6$ there the allowed charges are only electric charges associated with the $(20-2D)$ NS-NS sector abelian gauge fields.

VI. APPENDIX B: STATIC GENERATING SOLUTIONS

For the sake of simplicity we present explicitly only the explicit solution for the non-extreme *static* generating solution in $D = 4$, $D = 5$, $6 \leq D \leq 9$ and with four [29,30], three [3,26] and two charge [31] parameters of the NS-NS sector, respectively. Note that the full generating solution in $D = 4$ is parameterised by *five* charge parameters. For the explicit form of the rotating generating solution in $D = 5$ see [3] and in $6 \leq D \leq 9$ see [4,5], while in $D = 4$ the four charge parameter rotating solution is given in [8] and the five charge static solution is given in [3]. The parameterisation there is given in terms of the “toroidal” sector of toroidally compactified heterotic string.

We choose to parameterise the generating solutions in terms of the mass m of the D -dimensional Schwarzschild solution, and the boost parameters δ_i , specifying the charges of the solution. The notation used is similar to that in [32].

A. $D = 4$ -Four Charge Static Solution

The expression for the non-extreme dyonic rotating black hole solution in terms of the (non-trivial) four-dimensional bosonic fields is of the following form ⁴:

$$\begin{aligned} ds_E^2 &= -\lambda f dt^2 + \lambda^{-1}[f^{-1}dr^2 + r^2 d\Omega_2^2], \\ G_{11} &= \frac{T_1}{T_2}, \quad G_{22} = \frac{F_2}{F_1}, \quad e^{2\varphi} = \frac{F_1 F_2}{T_1 T_2}, \end{aligned} \quad (46)$$

where ds_E^2 specifies the Einstein-frame (D -dimensional) space-time line element, G_{ij} correspond to the internal toroidal metric coefficients and φ is the D -dimensional dilaton field (see Appendix A). Other scalar fields are constant and assume canonical values (one or zero). Here

$$f = 1 - \frac{2m}{r}, \quad \lambda = (T_1 T_2 F_1 F_2)^{-\frac{1}{2}} \quad (47)$$

and

⁴The four-dimensional Newton’s constant is taken to be $G_N^{D=4} = \frac{1}{8}$ and we follow the convention of [33], for the definitions of the ADM mass, charges, dipole moments and angular momenta.

$$\begin{aligned}
T_1 &= 1 + \frac{2m \sinh^2 \delta_{e1}}{r}, & T_2 &= 1 + \frac{2m \sinh^2 \delta_{e2}}{r}, \\
F_1 &= 1 + \frac{2m \sinh^2 \delta_{m1}}{r}, & F_2 &= 1 + \frac{2m \sinh^2 \delta_{m2}}{r},
\end{aligned} \tag{48}$$

The ADM mass, and four $U(1)$ charges Q_1, Q_2, P_1, P_2 , associated with the respective gauge fields $A_{1\mu}^{(1)}, A_{1\mu}^{(2)}, A_{2\mu}^{(1)}, A_{2\mu}^{(2)}$, can be expressed in terms of m , and four boosts $\delta_{e1, e2, m1, m2}$ in the following way:

$$\begin{aligned}
M &= 4m(\cosh^2 \delta_{e1} + \cosh^2 \delta_{e2} + \cosh^2 \delta_{m1} + \cosh^2 \delta_{m2}) - 8m, \\
Q_1 &= 4m \cosh \delta_{e1} \sinh \delta_{e1}, & Q_2 &= 4m \cosh \delta_{e2} \sinh \delta_{e2}, \\
P_1 &= 4m \cosh \delta_{m1} \sinh \delta_{m1}, & P_2 &= 4m \cosh \delta_{m2} \sinh \delta_{m2}.
\end{aligned} \tag{49}$$

The BH entropy is of the form [29,30]:

$$S_{BH} = 2\pi(4m)^2 \cosh \delta_{e1} \cosh \delta_{e2} \cosh \delta_{m1} \cosh \delta_{m2}, \tag{50}$$

which in the BPS-saturated limit ($m \rightarrow 0$, $\delta_{e1, e2, m1, m2} \rightarrow \infty$ while keeping $Q_{1,2}, P_{1,2}$ finite) reduces to the form [9]:

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 P_1 P_2}. \tag{51}$$

In the case of the fifth charge parameter q added, the BH entropy of the BPS-saturated black holes becomes [10]:

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 P_1 P_2 - \frac{1}{4} q^2 (P_1 + P_2)^2}. \tag{52}$$

B. $D = 5$ -Three Charge Static Solution

The expression for the non-extreme dyonic rotating black hole solution in terms of the (non-trivial) five-dimensional bosonic fields is of the following form [3,26]⁵:

$$\begin{aligned}
ds_E^2 &= -\lambda^2 f dt^2 + \lambda^{-1} [f^{-1} dr^2 + r^2 d\Omega_3^2], \\
G_{11} &= \frac{T_1}{T_2}, & e^{2\varphi} &= \frac{\tilde{T}^2}{T_1 T_2},
\end{aligned} \tag{53}$$

with other scalars assuming constant canonical values. Here

$$f = 1 - \frac{2m}{r^2}, \quad \lambda = (T_1 T_2 \tilde{T})^{-\frac{1}{3}}, \tag{54}$$

and

⁵The five-dimensional Newton's constant is taken to be $G_N^{D=5} = \frac{2\pi}{8}$.

$$T_1 = 1 + \frac{2m\sinh^2\delta_{e1}}{r^2}, \quad T_2 = 1 + \frac{2m\sinh^2\delta_{e2}}{r^2}, \quad \tilde{T} = 1 + \frac{2m\sinh^2\delta_{\tilde{e}}}{r^2}. \quad (55)$$

The ADM mass, three charges $Q_{1,2}, \tilde{Q}$ associated with respective gauge fields $A_{1\mu}^{(1)}, A_{1\mu}^{(2)}$ and \tilde{A}_μ (the gauge field related to the two from field $B_{\mu\nu}$ by a duality transformation), are expressed in terms of m , and three boosts $\delta_{e1,e2,\tilde{e}}$ in the following way:

$$\begin{aligned} M &= 2m(\cosh^2\delta_{e1} + \cosh^2\delta_{e2} + \cosh^2\delta_{\tilde{e}}) - 3m, \\ Q_1 &= 2m\cosh\delta_{e1}\sinh\delta_{e1}, \quad Q_2 = 2m\cosh\delta_{e2}\sinh\delta_{e2}, \quad \tilde{Q} = 2m\cosh\delta_{\tilde{e}}\sinh\delta_{\tilde{e}}. \end{aligned} \quad (56)$$

The BH entropy is of the form [26]:

$$S_{BH} = 2\pi(2m)^{\frac{3}{2}} \cosh\delta_{e1} \cosh\delta_{e2} \cosh\delta_{\tilde{e}}, \quad (57)$$

which in the BPS-saturated limit ($m \rightarrow 0$, $\delta_{e1,e2,\tilde{e}} \rightarrow \infty$ with $Q_{1,2}, \tilde{Q}$ finite) reduces to the form [24,11]:

$$S_{BH} = 2\pi\sqrt{Q_1 Q_2 \tilde{Q}}. \quad (58)$$

C. $6 \leq D \leq 9$ -Two Charge Static Solution

The expression for the non-extreme dyonic rotating black hole solution in terms of the (non-trivial) five-dimensional bosonic fields is of the following form [31,34,4,5]⁶:

$$\begin{aligned} ds_E^2 &= -\lambda^{D-3} f dt^2 + \lambda^{-1} [f^{-1} dr^2 + r^2 d\Omega_{D-2}^2], \\ G_{11} &= \frac{T_1}{T_2}, \quad e^{2\varphi} = \frac{1}{T_1 T_2}, \end{aligned} \quad (59)$$

while other scalar fields are constant and assume canonical values. Here

$$f = 1 - \frac{2m}{r^{D-3}}, \quad \lambda = (T_1 T_2)^{\frac{1}{D-2}}, \quad (60)$$

and

$$T_1 = 1 + \frac{2m\sinh^2\delta_{e1}}{r^{D-3}}, \quad T_2 = 1 + \frac{2m\sinh^2\delta_{e2}}{r^{D-3}}. \quad (61)$$

The ADM mass, $U(1)$ charges Q_1, Q_2 associated with respective $A_{1\mu}^{(1)}, A_{2\mu}^{(2)}$ gauge fields, are expressed in terms of m , and two boosts $\delta_{e1,e2}$ in the following way:

$$M = \frac{\omega_{D-2} m}{8\pi G_D} [(D-3)(\cosh^2\delta_{e1} + \cosh^2\delta_{e2}) - (D-4)],$$

⁶The D -dimensional Newton's constant is taken to be $G_N^D = \frac{(2\pi)^{D-4}}{8}$.

$$Q_1 = \frac{\omega_{D-2}m}{8\pi G_D}(D-3)\cosh\delta_{e1}\sinh\delta_{e1}, \quad Q_2 = \frac{\omega_{D-2}m}{8\pi G_D}(D-3)\cosh\delta_{e2}\sinh\delta_{e2}, \quad (62)$$

where $\omega_{D-2} = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$.

The BH entropy is of the form:

$$S_{BH} = \frac{\omega_{D-2}}{2G_D}m^{\frac{D-2}{D-3}}\cosh\delta_{e1}\cosh\delta_{e2}, \quad (63)$$

which in the near-BPS-saturated limit ($Q_{1,2} \gg m$) reduces to the form:

$$S_{BH} = 4\pi\sqrt{\frac{1}{(D-3)^2}Q_1Q_2(2m)^{\frac{2}{D-3}}}. \quad (64)$$

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TABLES

D	Classical Duality $-U$	Maximal Compact Subgroup $-C_U$	Quantum Duality $-Q_U$
4	$E_{7(7)}$	$SU(8)$	$E_{7(7)}(\mathbf{Z})$
5	$E_{6(6)}$	$USp(8)$	$E_{6(6)}(\mathbf{Z})$
6	$SO(5, 5)$	$SO(5) \times SO(5)$	$SO(5, 5; \mathbf{Z})$
7	$SL(5, \mathbf{R})$	$SO(5)$	$SL(5, \mathbf{Z})$
8	$SL(3, \mathbf{R}) \times SL(2, \mathbf{R})$	$SO(3) \times U(1)$	$SL(3, \mathbf{Z}) \times SL(2, \mathbf{Z})$
9	$SL(2, \mathbf{R}) \times \mathbf{R}^+$	$U(1)$	$SL(2, \mathbf{Z})$

TABLE I. The classical and quantum duality symmetries [16] for toroidally compactified Type II string in $4 \leq D \leq 9$.